Glasgow Math. J. **45** (2003) 153–157. © 2003 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089502001106. Printed in the United Kingdom

ALMOST p-STRUCTURES ON VECTOR-BUNDLES

I. DIBAG
Department of Mathematics, Bilkent University, 06533 Bilkent, Ankara, Turkey
e-mail: dibag@fen.bilkent.edu.tr

(Received 27 November, 2001; accepted 28 March, 2002)

Abstract. For $p \geq 2$ we introduce the notion of an almost p-structure on vector-bundles which generalizes the notion of an almost-complex structure and investigate the existence of almost p-structures on spheres and complex projective spaces.

2000 *Mathematics Subject Classification.* 55R10, 57R22.

0. Introduction. In this note we generalize the notion of an almost-complex structure on a real vector-bundle; i.e. a fibrewise linear map J on a vector-bundle ξ such that $J^2 = -1$. For $p \geq 2$ we consider a fibrewise linear map J on ξ such that $J^p = (-1)^{p-1}$. For $p = 2$ this gives an almost-complex structure, but for $p > 2$ this does not suffice. Let $a_p = R[x]/(x^p - (-1)^{p-1})$. This turns the fibre ξ_x into an a_p-module. Since a_p is not a field it does not automatically follow that $\xi_x = a_p^k$ for some $k \in \mathbb{Z}^+$. We insert one more condition which guarantees this. We call such maps J almost p-structures. We then study the structure of a_p as an algebra and prove that

$$a_p = \begin{cases} \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} & (\frac{p}{2} - \text{factors } \mathbb{C}) \quad \text{if } p \text{ is even} \\ \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C} & (\frac{p-1}{2} - \text{factors } \mathbb{C}) \quad \text{if } p \text{ is odd.} \end{cases}$$

It follows from this that a vector-bundle of dimension n admits an almost p-structure iff $n = kp$ for some $k \in \mathbb{Z}^+$ and splits into a direct-sum of $\frac{p}{2}$ complex vector-bundles of dimension k if p is even and into a direct-sum of a real vector-bundle and $(\frac{p-1}{2})$-complex vector bundles of dimension k if p is odd. Using this criterion we solve completely the existence problem of almost p-structures on spheres and complex projective spaces. The only non-trivial almost p-structures on spheres (i.e. on non-parallelisable ones) is an almost 3-structure on S^{15} in addition to the almost-complex structures on S^2 and S^6. The only almost p-structures that exist on complex projective spaces is an almost 3-structure on $P_3(\mathbb{C})$ in addition to the almost-complex structures that exist on all complex projective spaces. For this we rely heavily on [1].

1. Almost p-structures. For $p \geq 2$ let J be a fibrewise linear map on a vector-bundle ξ over a topological space X such that $J^p = (-1)^{p-1}$.

DEFINITION 1.1. Let $a_p = R[x]/(x^p - (-1)^{p-1})$. Then $a_p = \{1, x, \ldots, x^{p-1}/x^p = (-1)^{p-1}\}$. The fibre ξ_x is an a_p-module, the module structure is given by $x^i v = J^i(v)$, $v \in \xi_x (0 \leq i \leq p-1)$.

DEFINITION 1.2. For $v \in \xi_x$ define $E(v)$ to be the subspace generated by $v, J(v), \ldots, J^{p-1}(v)$.

DEFINITION 1.3. We call $v \in \xi_x$ a cyclic vector iff dim $E(v) = p$, i.e. iff v, $J(v), \ldots, J^{p-1}(v)$ are linearly-independent. For $v \in \xi_x$ a cyclic-vector, $E(v) = a_p$. For $p = 2$ every non-zero vector is a cyclic vector.

DEFINITION 1.4. A fibrewise linear map J on a vector-bundle ξ is called an almost p-structure on ξ iff

(i) $J^p = (-1)^{p-1}$ and (ii) For every J-invariant proper subspace U of ξ_x there exists a cyclic vector $v \notin U$.

We deduce from (ii) that there exist cyclic vectors $v_1, \ldots, v_k$ such that $\xi_x = E(v_1) \oplus E(v_2) \oplus \cdots \oplus E(v_k)\, n = kp$ i.e. $n \equiv 0 \pmod p$ and $\xi_x = a_p^k$. For $p = 2$ condition (ii) is vacuous and condition (i) suffices to define an almost 2-(i.e. almost-complex) structure.

2. Algebraic structure of a_p. For p even let $\theta_k = \frac{(2k-1)}{p}\pi$ and $x_k = \frac{2}{p}(1 + \sum_{m=1}^{\frac{p}{2}-1} \cos(m\theta_k)(x^m - x^{p-m}))(1 \leq k \leq \frac{p}{2})$. Then $x_k^2 = x_k$, $x_k x_\ell = 0\,(k \neq \ell)$ and $\sum_{k=1}^{p/2} x_k = 1$. Thus $a_p = \oplus_{k=1}^{p/2} I_k$ where I_k is the ideal generated by x_k. The homomorphism $R[x] \to I_k$ has kernel $(x - e^{i\theta_k})(x - e^{-i\theta_k}) = x^2 - 2x\cos\theta_k + 1$ and this gives an isomorphism of algebras $\mathbb{C} = R[x]/(x^2 - 2x\cos\theta_k + 1) \xrightarrow{=} I_k$. Thus $a_p = \mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}(\frac{p}{2}\text{-factors})$.

For p odd let $\psi_k = \frac{2k\pi}{p}(0 \leq k \leq \frac{1}{2}(p-1))$. $x_0 = \frac{1}{p}(1 + x + \cdots + x^{p-1}) x_k = \frac{2}{p}(1 + \sum_{m=1}^{\frac{1}{2}(p-1)} \cos(m\psi_k)(x^m + x^{p-m})(1 \leq k \leq \frac{1}{2}(p-1))$. Then $x_k^2 = x_k$, $x_k x_\ell = 0 (k \neq \ell)$ and $\sum_{k=0}^{\frac{1}{2}(p-1)} x_k = 1$. Thus $a_p = \oplus_{k=0}^{\frac{1}{2}(p-1)} I_k$ where I_k is the ideal generated by x_k. The homomorphism $R[x] \to I_k$ has kernel (i) $(1-x)$ for $k = 0$ and (ii) $(x - e^{i\psi_k})(x - e^{-i\psi_k}) = x^2 - 2x\cos\psi_k + 1 (1 \leq k \leq \frac{1}{2}(p-1))$. We obtain algebra isomorphisms (i) $R = R[x]/(1-x) \xrightarrow{=} I_0$ and (ii) $\mathbb{C} = R[x]/(x^2 - 2x\cos\psi_k + 1) \xrightarrow{=} I_k (1 \leq k \leq \frac{1}{2}(p-1))$. Hence $a_p = \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}\,(\frac{1}{2}(p-1)$ factors $\mathbb{C})$.

3. Almost p-structures on real vector-bundles. Let ξ be a real vector-bundle of dimension n over a topological space x with an almost p-structure J. We know from Section 1 that $n \equiv 0 \pmod p$. Let $n = kp$. For $x \in X$, the fibre ξ_x is an a_p-module. Let $x_i \in a_p$ be the elements defined in Section 2 such that a_p is the direct-sum of the ideals generated by x_i. Define $\xi_i(x) = \{x_i \cdot v | v \in \xi_x\}$. Then $\xi_x = \oplus_i \xi_i(x)$ and if we define $\xi_i = \bigcup_{x \in X} \xi_i(x)$, ξ decomposes into $\xi = \oplus_i \xi_i$. If p is even E_i is a complex vector-bundle of dimension k for $1 \leq i \leq \frac{p}{2}$. If p is odd E_0 is a real vector-bundle and E_i is a complex vector-bundle of dimension k for $1 \leq i \leq (\frac{p-1}{2})$. The argument is reversible. Suppose p is even and $\xi = \oplus_{i=1}^{p/2} \xi_i$ for complex vector-bundles ξ_i. Let J_i be the almost-complex structure on ξ_i. Define $x_i \cdot v = J_i(v)$ for $v \in \xi_i$. Then the i^{th}-factor $\mathbb{C}$ in the direct-sum decomposition of a_p acts on ξ_i and this defines an action of a_p on ξ. An analogous argument holds in the case p odd. This leads to

THEOREM 3.1. *A vector-bundle ξ of dimension n over a topological space X admits an almost p-structure iff $n \equiv 0 \pmod p$ i.e. $n = kp$ and*

(i) *if p is even $\xi = \oplus_{i=1}^{p/2} \xi_i$ where ξ_i is a complex vector-bundle of dimension k.*

(ii) *if p is odd $\xi = \xi_0 \oplus \oplus_{i=1}^{\frac{1}{2}(p-1)} \xi_i$ where ξ_0 is a real vector-bundle and ξ_i is a complex vector bundle of dimension k.* $(1 \leq i \leq \frac{1}{2}(p-1))$.

4. Almost p structures on spheres. It is well known that the even spheres which admit almost-complex structures are S^2 and S^6. We search for almost p-structures on spheres for $p > 2$. The only non-trivial almost p-structure that we can find is an almost 3-structure on S^{15}. We rely heavily on [**1**] for machinery and details. Let $L_k = 2^{v_2(M_k)}$ be the 2-primary component of the Atiyah–Todd number i.e. $v_2(M_k) = \sup_{1 \le r \le k-1}(r + v_2(r))$. We note that almost p-structures on S^k exist for all p/k when S^k is parallelisable i.e. if $k = 1, 3, 7$ and call such almost p-structures trivial. We call an almost p-structure non-trivial if the sphere in question is not parallelisable.

PROPOSITION 4.1. *Let p and k be odd. The only non-trivial almost p-structure on S^{pk} is an almost 3-structure on S^{15}.*

Proof. By Theorem 3.1 (ii), S^{pk} admits an almost p-structure iff the fibration

1. $SO(pk+1)/SO(k) \times U(k) \times \cdots \times U(k) \xrightarrow{SO(pk)/SO(k)\times U(k)\times\cdots\times U(k)} S^{pk}$ admits a cross-section. Let's fix one $U(k)$. Since $SO(k)$ and all the other $U(k)$'s can be imbedded in this fixed $U(k)$, by using the idea of proof of [2, Theorem 27.16] we deduce that fibration 1 admits a cross-section iff the fibration

2. $SO(pk+1)/U(k) \xrightarrow{SO(pk)/U(k)} S^{pk}$; admits a cross-section. If $\frac{pk+1}{2}$ is odd the existence of a cross-section to fibration 2 implies the existence of a cross-section to the Stiefel fibration

3. $V_{pk+1,(p-2)k+1} = SO(pk+1)/SO(2k) \xrightarrow{V_{pk,(p-2)k} = SO(pk)/SO(2k)} S^{pk}$ i.e. $a(p-2)k$-frame on S^{pk}. Since $pk+1 \equiv 2 \pmod 4$, S^{pk} admits at most a 1-frame and thus $(p-2)k = 1$ or $p = 3, k = 1$. Since S^3 is parallelisable this is the only case when fibration 2 admits a cross-section when $\frac{pk+1}{2}$ is odd.

For $\frac{pk+1}{2} \le 4$ is even. $\frac{pk+1}{2} = 2, 4$, S^{pk} is parallelisable and fibration 2 admits a cross-section. For $\frac{pk+1}{2} > 4$ and is even we deduce from [1, Proposition 4.3] and the discussion following it that fibration 2 admits a cross-section iff $L_{\frac{1}{2}((p-2)k+1)}/(\frac{pk+1}{2})$.

We observe that $L_n > 4n$ for $n > 4$. To see this, note that $L_5 = 2^6 > 4.5$ and for $k \ge 6$, $L_k \ge 2^{k-1} > 4k$.

For $\frac{(p-2)k+1}{2} > 4$, $L_{\frac{(p-2)k+1}{2}} - (\frac{pk+1}{2}) > 4(\frac{(p-2)k+1}{2}) - (\frac{pk+1}{2}) = \frac{1}{2}(k(3p-8)+3) > 0$ i.e. $L_{\frac{(p-2)k+1}{2}} > (\frac{pk+1}{2})$ so $L_{\frac{(p-2)k+1}{2}} \nmid (\frac{pk+1}{2})$ and thus fibration 2 does not admit a cross-section. For $\frac{(p-2)k+1}{2} \le 4$, we disregard the cases $\frac{(p-2)k+1}{2} = 2, 4$ since $\frac{pk+1}{2}$ is odd in either case. Let $\frac{k(p-2)+1}{2} = 1, k = 1, p = 3, S^{pk} = S^3$ is parallelisable. $\frac{k(p-2)+1}{2} = 3$, $k(p-2) = 5$. Either $k = 1$ and $p = 7$ and $S^{pk} = S^7$ is parallelisable or $p = 3$, $k = 5$, $\frac{pk+1}{2} = 8$ and $L_3 = 8/8$ and we obtain an almost 3-structure on S^{15}.

LEMMA 4.2. *Let p/q. Then the existence of an almost q-structure on a vector-bundle implies the existence of an almost p-structure.*

COROLLARY 4.3. *The only almost p-structures on spheres for p even are the almost-complex structures on S^2 and S^6.*

Proof. By Lemma 4.2 if a sphere admits an almost p-structure for p even then it admits an almost-complex structure and hence the sphere in question is S^2 or S^6. Apart from the almost-complex structures on these spheres, S^6 may admit an almost 6-structure. It follows from the proof of Proposition 4.1 it is equivalent to the cross-sectioning of the fibration $V_{7,5}=SO(7)/U(1)\xrightarrow{V_{6,4}=SO(6)/U(1))} S^6$; i.e. the existence of a 4-frame on S^6 which is impossible.

LEMMA 4.4. *An almost p-structure does not exist on S^{pk} for p odd and k even.*

Proof. The existence of an almost p-structure implies the existence of a frame on the even dimensional sphere S^{pk} which is impossible.

We gather Proposition 4.1. Corollary 4.3 and Lemma 4.4. in a single Theorem.

THEOREM 4.5. *The only non-trivial almost p-structures that exist on spheres are the almost* 2-(i.e. *almost-complex*) *structures on S^2 and S^6 and the almost* 3-*structure on S^{15}.*

5. Almost *p*-structures on complex projective spaces.

PROPOSITION 5.1. *For $p > 2$ the only almost p-structure on complex projective spaces is an almost* 3-*structure on $P_3(\mathbb{C})$.*

Proof. Suppose $P_{n-1}(\mathbb{C})$ admits an almost p-structure for $p>2$. Then $2(n-1)=kp$. Let $\pi : S^{2n-1}\to P_{n-1}(\mathbb{C})$ be the projection. Since $T(S^{2n-1})=\pi^!(T(P_{n-1}(\mathbb{C})))\oplus 1$ the fibration

$$SO(2n)/\underbrace{U(k)\times\cdots\times U(k)}_{p/2}\to S^{2n-1}$$

or the fibration

$$SO(2n)/SO(k)\times\underbrace{U(k)\times\cdots\times U(k)}_{(\frac{p-1}{2})}\to S^{2n-1}$$

admits a cross-section depending on whether p is even or odd. By the proof of [**2**, Theorem 27.16], in either case the fibration $SO(2n)/U(k)\to S^{2n-1}$ admits a cross-section and L_{n-k}/n by [**1**, Proposition 4.3] and discussion following it. As in the proof of Proposition 4.1, $L_{n-k}>4(n-k)>n$ for $n>k+4$ and $n>4$. Hence $L_{n-k}\nmid n$ for $n=\frac{1}{2}kp+1>k+4$ i.e. for 1. $(\frac{1}{2}p-1)k>3$. This is always satisfied for $p>8$. For $p=8$, $(\frac{1}{2}p-1)k>3$ unless $k=1$ in which case $n=5$, $n-k=4$ and $L_4\nmid 5$.

For $p=7$, 1 is satisfied unless $k=1$. $kp=7$ is a contradiction since kp is even. For $p=6$, 1 is satisfied unless $k=1$ in which case $n=4$. The existence of an almost 6-structure on $P_3(\mathbb{C})$ means that $(T(P_3(\mathbb{C}))$ is the direct-sum of three $U(1)$-bundles ξ_i. $(i=1,2,3)$. $T(P_3(\mathbb{C}))\oplus 1=4\eta_3$ where η_3 is the complex Hopf bundle over $P_3(\mathbb{C})$. Taking Pontryagin classes, $p(P_3(\mathbb{C}))=(1+y^2)^4$ where $y\in H^2(P_3;\mathbb{Z})$ is the generator. Suppose ξ_i has Pontryagin class $1+m_i^2y^2$, $m_i\in\mathbb{Z}$. Equating $(1+y^2)^4=\prod_{i=1}^3(1+m_i^2y^2)$. Hence $m_1^2+m_2^2+m_3^2=4$ which has solution $m_1=2$ and $m_2=m_3=0$. i.e. ξ_2 and ξ_3 are trivial. This implies the existence of a frame on $P_3(\mathbb{C})$ which is impossible.

For $p=5$ again we consider $k=1$ (otherwise 1 is satisfied). We disregard this case since kp should be even.

For $p=4$ and $k=1,2$. Let $k=2$, $n=5$, $L_3=8\nmid 5$. Let $k=1$, $n=3$ $L_2=2\nmid 3$. For $p=3$ since pk is even $k=2,4$. Let $k=4$, $n=7$, $L_3\nmid 7$ $k=2$, $n=4$. Let $\tau : P_3(\mathbb{C})\to P_1(Q)$ be the projection onto the one dimensional quaternionic projective space. Let J be the quaternionic structure on $\mathbb{C}^4$ which anti-commutes with the complex structure. The assignment $x\mapsto J(x)(x\in S^7)$ defines a unit vector-field on $\pi^!(T(P_3(\mathbb{C})))$ and passes to the quotient and generates a line sub-bundle ξ of $T(P_3(\mathbb{C}))$ whose orthogonal complement is $\tau^!(T(P_1(\mathbb{Q})))$. Hence $\tau^!(T(P_1(\mathbb{Q})))$ admits an almost-complex structure and $T(P_3(\mathbb{C})) = \xi \oplus \tau^!(T(P_1(\mathbb{Q})))$ an almost 3-structure. $\square$

REFERENCES

1. I. Dibag, Almost complex substructures on the sphere, *Proc. Amer. Math. Soc.* **61** (2) (1976), 361–366.

2. N. E. Steenrod, The topology of fibre bundles (Princeton University Press, N.J., 1951).